論文　**Original Paper**

d-primitive preserving morphisms

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Abstract: Let Q be the set of all primitive words over a finite alphabet having at least two letters. In this paper, we study the language $D(1)$ of all non-overlapping (d-primitive) words, which is a proper subset of Q, and we present a necessary and sufficient condition for a morphism to preserve the d-primitivity.

Key words: primitive word, non-overlapping word, d-primitive word, morphism

1. Introduction

The notion of primitivity of words play a central role in algebraic coding theory and combinatorial theory of words (See [4] [5], and [7]).

Recently attention has been drawn to the language Q of all primitive words. Some studies on a problem concerning the behavior of morphims with respect to primitivity has been done. A sufficient condition for a morphism to preserve the primitivity has been presented in [9], and classification of morphisms from the point of view of their primitivity-preserving properties has been done in [6].

In this paper, we study the language $D(1)$ of non-overlapping (d-primitive) words, which is a proper subset of Q, and we present a necessary and sufficient condition for a morphism to preserve the d-primitivity. In Section 2, some basic definition and results are presented. In Section 3, we present a necessary and sufficient condition of a morphism to be d-primitive preserving.

2. Preliminaries

Let Σ be an alphabet consisting of at least two letters. Σ^* denotes the free moniod generated by Σ, that is, the set of all finite words over Σ, including the empty word ε, and $\Sigma^+ = \Sigma^* - \{\varepsilon\}$. For w in Σ^*, $|w|$ denotes the length of w, A *language* over Σ is a set $L \subseteq \Sigma^*$.

For a word $u \in \Sigma^+$, if $u = vw$ for some $v, w \in \Sigma^*$, then v (w) is called a *prefix (suffix)* of u, denoted by $v \leq_p u$ ($w \leq_s u$, *resp*.). If $v \leq_p u$ ($w \leq_p u$) and $u \neq v$ ($w \neq u$), then v (w) is called a *proper prefix* (*proper suffix*) of u, denoted by $v <_p u$ ($w <_p u$, resp.). For a word w, let $Pref(w)$ ($Suff(w)$) be the set of all prefixes (suffixes, resp.) of w.

A nonempty word u is called a *primitive word* if $u = f^n$, $f \in \Sigma^+$, $n \geq 1$ always implies that $n = 1$. Let Q be the set of all primitive words over Q. A nonempty word u is a *non-overlapping word* if $u = vx = yv$ for $x, y \in \Sigma^+$ always implies that $v = \varepsilon$. Let $D(1)$ be the set of all non-overlapping words over Σ. A word in $D(1)$ is also called a *d-primitive word* (See [1] and [8]).

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For $x, y \in \Sigma^+$, if $(Pref(x) - \{\varepsilon\}) \cap (Suff(y) - \{\varepsilon\}) = \phi$, then (x, y) is said to be a non-overlapping pair (n-o. pair).

Remark 1 *Let $u, v \in \Sigma^+$. Obviously $uv \in D(1)$ implies that (u, v) is a n-o. pair. The converse does not hold; for u = abbbba, and v = bb, (u, v) is a n-o. pair but uv is not in D(1). However, in the next Proposition, we show the above two are equivalent on the condition that u and v are in D(1),*

Proposition 1 *For $u \in \Sigma^+$, the following two are equivalent.*
(1) u, v, uv, and vu are in D(1).
(2) u, v are in D(1), and (u, v), (v, u) are n-o. pairs.

Proof.
(1) $\Rightarrow$ (2) : Obvious.
(2) $\Rightarrow$ (1) : Suppose that (2) holds but $uv \notin$ D(1) or $vu \notin D(1)$. It suffices to show that only the case for $uv \notin D(1)$. We can write $uv = zwz$ for some $z \in \Sigma^+$, $w \in \Sigma^*$. Since (u, v) is n-o.pair, obviously $|u| \neq |z|$.
(Case 1) $z <_p u$
(1.1) $zw <_p u$
We have that $u = zwy$ and $z = yv$ for some $y \in \Sigma^+$. Thus $u = yvwy \notin D(1)$.
(1.2) $u <_p zw$
We have that $u = zw_1$ and $v = w_2z$ for some $w_1, w_2 \in \Sigma^+$ with $w = w_1w_2$. Thus (u, v) is not n-o. pair.
(Case 2) $u <_p z$
We have that $v = xwz$, $z = ux$ for some $x \in \Sigma^+$. Thus $v = xwux \notin D(1)$. □

3. d-primitive preserving

Lemma 2 *([2]) Let $u \in \Sigma^+$. Then $u \notin D(1)$ iff there exists a unique word $v \in D(1)$ with $|v| \leq (1/2)|u|$ such that $u = vwv$ for some $w \in \Sigma^*$.*

A morphism $h : \{a, b\}^* \to \{a, b\}^*$ is said to be d-primitive preserving, if $h(x) \in D(1)$ for all $x \in D(1)$.

Proposition 3 *A morphism h is d-p.p. iff the following two conditions hold.*
(1)Both $h(a)$ and $h(b)$ are in $D(1)$. (2)Both $(h(a), h(b))$ and

$(h(b), h(a))$ *are nonoverlapping pairs.*

Proof.

[Only if] If $h(a) \notin D(1)$ or $h(b) \notin D(1)$, then obviously h is not d-p.p.. Suppose that $h(a) = wu$ and $h(b) = vw$ for some u, $v \in \Sigma^*$ and $w \in \Sigma^+$. Then $h(ab) = wuvw \notin D(1)$. Thus h is not a d-p.p. morphism.

[If] Suppose that both (1) and (2) hold but that a morphism h is not d-p.p.. There exist $u, v \in \Sigma^+$, and $w \in \Sigma^*$ such that $u \in D(1)$, and $h(u) = vwv$. Let $u = v_1v_2...v_k$ for some $v_i \in \Sigma$. If $v = h(v_1...v_i) = h(v_j...v_k)$ for some i and j, then $j + i - 1 = k$, and $v_1 = v_j$, $v_2 = v_{(j+1)}$, ..., $v_i = v_k$. This means that $u \notin D(1)$.

We can assume that $v \notin (h(\Sigma))^*$. Suppose that $v \in (h(\Sigma))^* = h(\Sigma^*)$. Let $u = v_1...v_k\eta v'_1 ...v'_m$ for some $v_i, v'_j \in \Sigma$ and $\eta \in \Sigma^*$, where $h(\eta) = w$, $h(v_i) = v_i$, $h(v'_j) = v'_j$, $1 \le i \le k$, $1 \le j \le m$, $v = v_1...v_k = v'_1 ...v'_m$. We can assume that $v = v'w'$ and $v = v''y$ for some $w' \in Pref(h(\Sigma))$ and $y \in h(\Sigma)$, $v', v'' \in \Sigma^*$.

If $|w'| \le |y|$, then condition(2) dose not hold. Thus we can write $w' = xy$ for some $x \in \Sigma^+$. We can assume that $h(a) = xyz$ and $h(b) = y$ for some $z \in \Sigma^+$.

(Case 1-1) $z = sx$; $|x| < |z|$, $s \in \Sigma^+$.

$h(a) = xyz = xysx$, contradicting (1).

(Case 1-2) $x = y''z$; $|z| < |x| < |yz|$, with $y = y'y''$, for $y'' \in \Sigma^+$.

$h(a) = y''zyz$, $h(b) = y = y'y''$, contradicting (2).

(Case 1-3) $x = x''yz$; $|yz| < |x|$, $x = x'x''$, $x'' \in \Sigma^+$.

$h(a) = xyz = x''yzyz = x'x''yz$, contradicting (1).

(Case 2-1) $x = x''yzyn$ for some $x', x'' \in \Sigma^+$ with $x = x'x''$, and $n \ge 1$.

h(a) = $xyz = x''yzynyz = x'x''yz$, contradicting (1).

(Case 2-2) $x = y''yn$ for some $y'', y' \in \Sigma^+$ with $y = y'y''$, and $n \ge 1$. We have that $h(a) = xyz = y''ynyz$ and $h(b) = y'y''$, contradicting (2). □

Corollary 4 *A morphism* h: $\{a, b\}^* \rightarrow \{a, b\}^*$ *is d-p.p. if and only if four words* $h(a)$, $h(b)$, $h(a)h(b)$ *and* $h(b)h(a)$ *are in* $D(1)$.

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